Math 656 • FINAL EXAM • May 13, 2014

1) (8pts) Find **all values** of $\tanh^{-1}(i)$.

$$\tanh z = i = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z}-1}{e^{2z}+1} \;\Rightarrow\; i\left(e^{2z}+1\right) = e^{2z}-1 \;\Rightarrow e^{2z}\left(1-i\right) = 1+i \;\Rightarrow\; e^{2z} = \frac{1+i}{1-i} = i$$

$$\Rightarrow\; 2z = \log i = \frac{i\pi}{2} + i2\pi k \;\Rightarrow \boxed{z = i\pi\left(\frac{1}{4}+k\right),\quad k \in \mathbb{Z}}$$

2) (12pts) Categorize **all singularities** of the following functions. Examine also a possible singularity at z=∞ (hint: substitute $\zeta = 1/z$). Make sure to explain briefly.

(a) $f(z) = \dfrac{1}{z^{1/4}\sin z} = \dfrac{\zeta^{1/4}}{\sin(1/\zeta)}$
$$\begin{cases} z=0: & \text{Branch point} \\ z=\infty: & \text{Branch point / cluster point} \\ z=\pi k,\ k\in\mathbb{Z},\ k\neq 0: & \text{Simple zeros} \end{cases}$$

(b) $f(z) = \dfrac{\exp(z)}{\exp(1/z)} = \exp\left(z - \dfrac{1}{z}\right) = \exp\left(\dfrac{1}{\zeta} - \zeta\right)$ Essential singularity at $z = 0$ and at $z = \infty$

(c) $f(z) = \dfrac{\sin(\pi z)}{\sin^2(\pi/z)} = \dfrac{\sin(\pi/\zeta)}{\sin^2(\pi\zeta)}$
$$\begin{cases} z=0: & \text{Cluster point} \\ z=\infty: & \text{Essential singularity} \\ z=\pm 1: & \text{Simple poles} \\ z=\dfrac{1}{k},\ k\in\mathbb{Z},\ |k|\geq 2: & \text{Poles of order 2} \end{cases}$$

3) (12pts) Find the first **two** dominant terms in the series expansion of $f(z) = \dfrac{\cos\left(\log_p(z)\right)-1}{\sin \pi z}$ around $z = 1$.
Hint: a shift $z = 1+\zeta$ may help. What would be the radius of convergence of the full series around z=1?

$$f(z) = \frac{\cos\left(\log_p(z)\right)-1}{\sin \pi z} = \frac{\cos\left(\log_p(1+\zeta)\right)-1}{\underbrace{\sin \pi(\zeta+1)}_{-\sin \pi\zeta}} = \frac{\cos\left(\zeta - \frac{\zeta^2}{2} + O(\zeta^3)\right)-1}{-\left(\pi\zeta + O(\zeta^3)\right)}$$

$$= \frac{-\frac{1}{2!}\left(\zeta - \frac{\zeta^2}{2}\right)^2 + O(\zeta^4)}{-\pi\zeta\left(1+O(\zeta^2)\right)} = \frac{\zeta}{2\pi}\,\frac{\left(1-\frac{\zeta}{2}\right)^2 + O(\zeta^2)}{1+O(\zeta^2)} = \frac{\zeta}{2\pi}\,\frac{1-\zeta+O(\zeta^2)}{1+O(\zeta^2)} = \boxed{\frac{\zeta}{2\pi}\left(1-\zeta\right)+O(\zeta^3)}$$

$$\Rightarrow\; f(z) = \frac{1}{2\pi}\left[(z-1)-(z-1)^2 + O\left((z-1)^3\right)\right]$$ Convergence radius = distance to singularity = 1: $|z-1|<1$

4) (16pts) Calculate the following integrals, picking the most efficient method for each. Contours are circles of given radius:

(a) $$\oint_{|z|=1}\frac{dz}{(\bar{z})^{1/4}}=\int_0^{2\pi}\frac{d\left(e^{i\theta}\right)}{\left(e^{-i\theta}\right)^{1/4}}=\int_0^{2\pi}\frac{ie^{i\theta}d\theta}{e^{-i\theta/4}}=i\int_0^{2\pi}e^{i\frac{5\theta}{4}}d\theta=\frac{4}{5}\left[\exp\left(i\frac{5\theta}{4}\right)\right]_0^{2\pi}=\frac{4}{5}\left(e^{i\frac{5\pi}{2}}-1\right)=\boxed{\frac{4}{5}(i-1)}$$

(b) Use mapping $z=\frac{1}{\zeta}$: $$\oint_{|z|=2}\frac{\exp(1/z)}{1-z^2}dz=\oint_{|\zeta|=\frac{1}{2}}\frac{\exp(\zeta)\,d\zeta}{\zeta^2\left(1-\frac{1}{\zeta^2}\right)}=\oint_{|\zeta|=\frac{1}{2}}\underbrace{\frac{\exp(\zeta)}{\zeta^2-1}}_{\text{Analytic wihin contour}}d\zeta=\boxed{0}$$

Note: without the mapping the solution is longer: $$=2\pi i\left(\underbrace{\underbrace{\text{Res}(-1)}_{1/(2e)}+\underbrace{\text{Res}(+1)}_{-e/2}}_{-\sinh 1}+\underbrace{\text{Res}(0)}_{1+\frac{1}{3!}+\frac{1}{5!}+\ldots=\sinh 1}\right)=0$$

5) (16pts) Calculate the following two integrals. Carefully explain each step, and make sure to obtain a **real** answer.

(a) $$\oint\frac{dz}{z^{1/2}(z+1)}=\int_0^\infty\frac{dr}{\sqrt{r}(r+1)}+\int_{z=re^{i2\pi}}\frac{dz}{z^{1/2}(z+1)}+\underbrace{\int_{C_R^+}\frac{dz}{z^{1/2}(z+1)}}_{\underbrace{|\ldots|\le\frac{2\pi R}{\sqrt{R}(R-1)}=O\left(\frac{1}{\sqrt{R}}\right)}_{\to 0\text{ as }R\to\infty}}+\underbrace{\int_{C_\varepsilon^-}\frac{dz}{z^{1/2}(z+1)}}_{\underbrace{|\ldots|\le\frac{2\pi\varepsilon}{\sqrt{\varepsilon}(1-\varepsilon)}=O(\sqrt{\varepsilon})}_{\to 0\text{ as }\varepsilon\to 0}}=2\pi i\,\text{Res}(-1)=\frac{2\pi i}{\sqrt{-1}}=\boxed{2\pi}$$

Note that along the bottom of the branch cut $z=re^{i2\pi}\Rightarrow z^{1/2}=-r$

Take the limit $\varepsilon\to 0,\ R\to\infty$: $$\int_0^\infty\frac{dr}{\sqrt{r}(r+1)}-\int_\infty^0\frac{dr}{\sqrt{r}(r+1)}=2I=2\pi\Rightarrow\boxed{I=\pi}$$

(b) $I\equiv\int_0^{+\infty}\frac{x^3\,dx}{a^6+x^6}$ Integrate around appropriate circular sector ("wedge") containing only one pole

Integral over the smallest wedge both sides of which satisfy $z^6=r^6$: consider a ray $z=re^{i2\pi/6}=re^{i\pi/3}$

$$\oint\frac{z^3\,dz}{a^6+z^6}=\int_0^\infty\frac{r^3dr}{a^6+r^6}+\int_\infty^0\frac{\overbrace{\left(e^{i\pi/3}\right)^3}^{-1}\left(e^{i\pi/3}dr\right)}{a^6+r^6}+\int_{|z|=R}\frac{z^3\,dz}{a^6+z^6}$$

$$=\left(1+e^{i\pi/3}\right)\int_0^R\frac{r^3\,dr}{a^6+r^6}+\underbrace{\int_{|z|=R}\frac{z^3\,dz}{a^6+z^6}}_{|\ldots|\le\frac{R^3\pi R/3}{R^6-a^6}=O\left(\frac{1}{R^2}\right)}=2\pi i\underbrace{\text{Res}\left(ae^{i\pi/6}\right)}_{\frac{N(z_0)}{D'(z_0)}}=\frac{2\pi i\left(ae^{i\pi/6}\right)^3}{6\left(ae^{i\pi/6}\right)^5}=\frac{\pi i}{3a^2}e^{-i\pi/3}=\frac{\pi e^{i\pi/6}}{3a^2}$$

Take the limit $R\to\infty$: $$\left(1+e^{i\pi/3}\right)I=\frac{\pi e^{i\pi/6}}{3a^2}\Rightarrow I=\frac{\pi}{3a^2}\frac{e^{i\pi/6}}{1+e^{i\pi/3}}\cdot\frac{e^{-i\pi/6}}{e^{-i\pi/6}}=\frac{\pi}{3a^2}\frac{1}{\underbrace{e^{i\pi/6}+e^{-i\pi/6}}_{2\cos(\pi/6)=\sqrt{3}}}=\boxed{\frac{\pi}{3\sqrt{3}a^2}}$$

6) (12pts) Use Rouche's Theorem to find the number of zeros of $f(z) = 4z^4 + 13z^2 + 3$ belonging to the following domains: (a) $|z| < 1$; (b) $|z| < 2$; (c) $1 < |z| < 2$

- Two roots inside $|z| < 1$:

  Consider $f(z) = 13z^2$ (which has two roots inside $|z|=1$) and $g(z) = 4z^4 + 3$

  On the circle $|z|=1$ we have $|g| \le 4|z^4| + 3 = 7 \quad \Rightarrow \quad |g| < |f| = 13|z|^2 = 13$

- All four roots inside $|z|<2$:

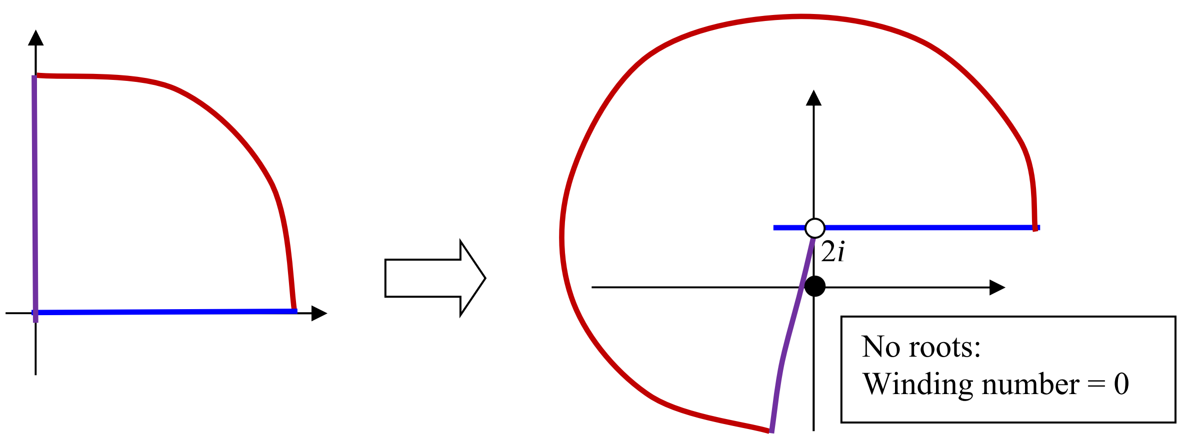
  Consider $f(z) = 4z^4$ (which has 4 roots inside $|z|=2$) and $g(z) = 3 + 13z^2$

  On the circle $|z|=2$ we have $|g| \le 3 + 13|z|^2 = 3 + 52 = 55 \quad \Rightarrow \quad |g| < |f| = 4 \cdot 2^4 = 64$

- Therefore, there are 4-2=2 roots in the ring (annulus) $1<|z|<2$

Do two of the last four problems:

7) (12pts) Use the Argument Principle to find the number of roots of $f(z) = 2i - z + z^2 + z^3$ lying in the first quadrant. To do this, sketch the mapping of the relevant quarter-circle (it's quite straightforward).



- Mapping of positive real axis: $f(z) = \underbrace{-x + x^2 + x^3}_{u} + \underbrace{2}_{v} i \quad \Rightarrow \quad v = 2 = const \quad \Rightarrow$ horizontal line v=2

- Mapping of quarter-circle:

  $R \to \infty \;\Rightarrow\; f(Re^{i\theta}) \approx (Re^{i\theta})^3 = R^3 e^{3i\theta},\;\; \theta \in [0,\; \pi/2) \;\Rightarrow$ Approaches 3/4 of a circle as $R \to \infty$

- Mapping of the imaginary axis, $z = iy$ where $y \ge 0$ (the only non-trivial and crucial part):

$$f(z = iy) = \underbrace{-y^2}_{u} + i\underbrace{\left(2 - y - y^3\right)}_{v}$$

$$\left\{\begin{array}{l} v = \operatorname{Im} f\left(z = iy\right) \text{ monotonically decreasing (curve bends downward as } y \to \infty) \\ u = \operatorname{Re} f\left(z = iy\right) \text{ also monotonically decreasing (bends to the left as } y \to \infty) \end{array}\right. \Rightarrow \text{curve bends away from origin}$$

8) **(12pts)** Suppose $f(z)$ is an entire function, satisfying inequality $|f(z)| < a + |z|^k$ everywhere in the complex plane (here $a > 0$ is a real constant). Prove that $f(z)$ is a polynomial. Hint: recall the proof of the Liouville's Theorem using the extended version of Cauchy Integral Formula.

Apply Cauchy Integral Formula to the (k+1)st derivative, and take a circle as the contour; since the function is entire, we can increase the circle size to infinity:

$$\frac{d^{k+1}f}{dz^{k+1}}(z) = \frac{k!}{2\pi i}\oint_{C_R}\frac{f(\zeta)\,d\zeta}{(\zeta - z)^{k+2}}$$

$$\Rightarrow \left|\frac{d^{k+1}f}{dz^{k+1}}(z)\right| \le \frac{k!}{|2\pi i|}\oint_{|\zeta|=R}\frac{|f(\zeta)||d\zeta|}{\left|(\zeta - z)^{k+2}\right|} = \frac{k!}{2\pi}\oint_{|\zeta|=R}\frac{|f(\zeta)|R\,d\theta}{R^{k+2}}$$

$$< \frac{k!}{2\pi}\oint_{|\zeta|=R}\frac{\left(a+R^k\right)R\,d\theta}{R^{k+2}} = k!\left(\frac{a}{R^{k+2}}+\frac{1}{R}\right) \to 0 \text{ as } R\to\infty$$

Thus, all derivatives of order k+1 and higher are zero, which means that the Taylor series has finitely many terms $\Rightarrow$ it's a polynomial

9) **(12pts)** Indicate domains of convergence of each series:

a) $\displaystyle\sum_{k=0}^{\infty}\frac{\exp(2zk)}{k!} = \sum_{k=0}^{\infty}\frac{\left[\exp(2z)\right]^k}{k!} = \exp\left(\exp(2z)\right)$ $\boxed{\text{Converges in entire } \mathbb{C}}$ (Also follows from the ratio test)

b) $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{\exp(-zk)}{k} = \sum_{k=1}^{\infty}(-1)^k\frac{\left[\exp(-z)\right]^k}{k} = -\log\left(1-e^{-z}\right)$

(You can also use the ratio test) Converges wherever $|e^{-z}| = e^{-x} < 1 \Rightarrow x > 0 \Rightarrow$ $\boxed{\text{Right half-plane}}$

10) **(12pts)** Consider the map $w = z + \dfrac{1}{z}$. Describe the images of the following sets under this map: **(a)** unit circle |z|=1, **(b)** circle of radius 2, |z|=2. **(c)** exterior of the unit disk, |z|>1. Hint: examine Cartesian components of the image, $w = u + i\,v$

(a) Unit cirlce: $z = e^{i\theta} \Rightarrow w = z + \dfrac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta \Leftarrow$ $\boxed{\text{Segment of real axis, } x \in [-2,\,2]}$

(b) Circle $|z| = 2$: $z = 2e^{i\theta} \Rightarrow w = 2e^{i\theta} + \dfrac{1}{2}e^{-i\theta} = \dfrac{5}{2}\cos\theta + i\dfrac{3}{2}\sin\theta \Leftarrow$ $\boxed{\text{Equation of ellipse } \left(\frac{x}{5/2}\right)^2 + \left(\frac{y}{3/2}\right)^2 = 1}$

(c) This mapping is conformal on $\mathbb{C}\backslash\{0,\infty\}$ $\Rightarrow$ Interiors map to interiors, boundaries map to boundaries

$\Rightarrow$ Both interior and exterior of unit circle map to the entire $\mathbb{C}$ excluding segment of real axis with $\operatorname{Re} z \in [-2,\,2]$

(The only domain that has a boundary which is a line segment is the entire $\mathbb{C}$ plane lying ouside of this line segment)